MATH 2850: CAUCHY EULER EQUATIONS

DEFINITION: A **Cauchy Euler** DE is a DE of the form: $ax^2y'' + bxy' + cy = 0$. We assume $x > 0$.

STRATEGY: Try for solutions of the form $y = x^m$. Then $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Hence:

$$ax^2y'' + bxy' + cy = 0 \implies ax^2m(m-1)x^{m-2} + bxmx^{m-1} + cx^m = 0 \implies am(m-1)x^m + bmx^m + cx^m = 0$$

Factoring gives: $x^m\left[am(m-1) + bm + c\right] = 0$. Assuming $x > 0$, we get: $am(m-1) + bm + c = 0$.

EXAMPLE: Solve $x^2y'' + 2xy' - 2y = 0$ for $x > 0$.

Ans: $y = c_1x + c_2x^{-2}$

EXAMPLE: Use Variation of Parameters to solve: $x^2y'' + 2xy' - 2y = 4x^2 + 3x$, $y(1) = 0$, $y'(1) = 11$

Ans: $y = 2x - 3x^{-2} + x^2 + x\ln(x)$

EXAMPLE: Solve: $x^2y'' - 5xy' + 9y = 0$ for $x > 0$.

NOTE: Use Reduction of Order to find a second, linearly independent solution.

Ans: $y = c_1x^3 + c_2x^3 \ln(x)$

EXAMPLE: Solve: $x^2y'' + xy' + 4y = 0$ for $x > 0$.

NOTE: Rewrite: $x^{bi} = e^{ib\ln(x)} = \cos(b\ln(x)) + i\sin(b\ln(x))\ \ldots$

Ans: $y = c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))$

EXAMPLE: Solve $x^2y'' + 5xy' + 5y = 0$ for $x > 0$.

NOTE: Rewrite: $x^{a+bi} = x^a x^{ib} = x^a e^{ib\ln(x)} = x^a\left[\cos(b\ln(x)) + i\sin(b\ln(x))\right]\ \ldots$

Ans: $y = x^{-2}\left[c_1\cos(\ln(x)) + c_2\sin(\ln(x))\right]$

SUMMARY: If the solutions to the auxiliary equation for the DE $a_2x^2y'' + a_1xy' + a_0y = 0$ is/are

- two distinct real numbers m_1 and m_2, the general solution is $y = c_1x^{m_1} + c_2x^{m_2}$.
- one repeated real number m, the general solution is $y = c_1x^m + c_2x^m \ln(x)$.
- two purely imaginary numbers $\pm bi$, the general solution is $y = c_1 \sin(b\ln(x)) + c_2 \cos(b\ln(x))$.
- two nonzero complex numbers $a \pm bi$, the general solution is $y = x^a\left[c_1 \sin(b\ln(x)) + c_2 \cos(b\ln(x))\right]$

For non-homogeneous Cauchy Euler equations, use Variation of Parameters.

BONUS TRACKS: Using substitutions to solve Cauchy Euler Equations:

Let $x = e^t$ so $t = \ln(x)$ so that $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{1}{x}\dfrac{dy}{dt}$ and:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{1}{x}\frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{1}{x}\frac{d}{dt}\left[\frac{1}{x}\frac{dy}{dt}\right] = \frac{1}{x}\left[-\frac{1}{x^2}\frac{dx}{dt}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\right] = \frac{1}{x}\left[-\frac{1}{x^2}x\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\right]$$

That is, substitute

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt} \text{ and } \frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

into $ax^2y'' + bxy' + cy = 0$ to transform the DE into a constant coefficient ODE!

EXAMPLE: Use the above technique to solve: $x^2y'' - 5xy' + 9y = 0$ for $x > 0$.

HOMEWORK: Pg. 347: 1 - 17 odd